LIKE LOVE, ALGEBRA IS WHERE YOU FIND IT. You can locate it almost anywhere in the middle school curriculum if you know where to look and what to look for. But with so many demands on our time, we often forget to look. We take problems at face value, and we assume that a geometry problem is just a geometry problem or that a data analysis activity is only about data analysis. If we scratch below the surface, however, we can find rich opportunities for algebraic thinking lurking in number explorations, measurement tasks, and geometry investigations.

Today, many school districts require students to begin the formal study of algebra in the eighth grade. Consequently, a major component of the early middle school years is dedicated to fostering algebraic thinking, yet middle school students are also expected to learn many concepts and skills that are not directly related to algebra. According to Principles and Standards for School Mathematics (NCTM 2000), the Algebra Standard should receive about one-third of the emphasis in the middle grades. (A graphic on page 30 of Principles and Standards “shows roughly how the Content Standards might receive different emphases across the grade bands.” This graphic is not meant to dictate the specific number of lessons for each topic throughout the year, but it does suggest that the Algebra Standard should receive approximately 30 percent of the emphasis in the middle grades.)

With so little time reserved for the Algebra Standard in the early middle school years, how is it possible to foster algebraic thinking while also sufficiently addressing the Standards of Number and Operations, Geometry, Measurement, and Data Analysis and Probability?

One effective approach is to use rich tasks that address more than one Standard. Principles and Standards states that the Content Standards do “not neatly separate the school mathematics curriculum into nonintersecting subsets. Because mathematics as a discipline is highly interconnected, the areas described by the Standards overlap and are integrated” (NCTM 2000, p. 30).

Teachers often choose rich activities for the classroom that allow for the exploration of topics from more than one standard. However, when they fail to highlight the connections between the Standards, teachers may lose an opportunity to promote algebraic skills in the middle grades. This article identifies some of the algebra implicit within activities that emphasize other concepts, connects traditional algebra problems to the other four Standards, and offers strategies for modifying activities so that they will be able to foster algebraic thinking.
What Is Algebraic Thinking?

MENTION THE WORD "ALGEBRA" AND MANY people—students and teachers included—think of x’s and equations and manipulating variables according to a set of rules. Although symbolic manipulation is arguably one of the most important parts of algebra, it is not the only part. Algebraic thinking involves much more.

Algebra is often described as the study of generalized arithmetic. Instead of dealing solely with number and computation, it focuses on operations and processes. Greenes and Findell suggest that the big ideas of algebraic thinking involve representation, proportional reasoning, balance, meaning of variable, patterns and functions, and inductive and deductive reasoning (Greenes and Findell 1998). Similarly, Kriegler contends that algebraic thinking combines two very important components: algebraic ideas, including patterns, variables, and functions, with mathematical thinking tools, specifically problem solving, representation (e.g., diagrams, words, tables), and reasoning (Kriegler 2004). In short, algebraic thinking is used in any activity that combines a mathematical process with one of the big ideas in algebra, such as understanding patterns and functions, representing situations with symbols, using mathematical models, and analyzing change. As such, opportunities for algebraic thinking arise in many contexts, as illustrated by the examples that follow.

Algebraic Thinking in a Number Sense Task

THE PLACE-VALUE NUMERATION SYSTEM IMPLICITLY incorporates some of the basic concepts of algebra, and the algorithms of number operations rely heavily on the “laws of algebra” (National Research Council 2001, p. 256). It is not surprising, then, that many middle-grades activities dealing with number and operations involve algebraic thinking, even if the task does not specifically target patterns, variables, or other algebraic ideas.

One classroom activity that combines number sense and algebraic thinking is the “positively negative” game, taken from Operating with Integers (ETS 2003c); the rules for the game are given in figure 1. This stimulating activity involves the multiplication and addition of positive and negative integers; open an introductory algebra textbook and this topic usually appears on the first few pages. As students play the game, they are often unaware that they are practicing basic operations and developing algebraic ideas. The task subtly improves students’ skills in mathematics.

The following is an example of a question that fosters algebraic thinking. It asks students to generate a sequence of scores for the “positively negative” game that would lead to a particular outcome.

Fig. 1  The rules for the “positively negative” game

On each turn, a player spins the two spinners shown below. The score for the turn is found by taking the product of the numbers on which the spinners stop. For example, if the first spinner stops on -5 and the second spinner stops on 2, the score for that turn would be -5 x 2, or -10.

A running total is kept by adding the score for each turn, and the player with the highest total after 10 turns is the winner.
After Jacob and Simone had each taken 7 turns, Jacob was leading 63 to 12. Simone could not believe that she had scored only 12 points! Show a sequence of scores that would yield a total score of 12 points after 7 turns. (Adapted from ETS 2003c, p. 11)

This question invokes algebraic thinking, because it combines the process of problem solving with the algebraic idea of patterns. It also forces students to think about issues of balance—one of the big ideas listed by Greenes and Findell—by considering additive inverses, and how positive values can be canceled by negative values.

When the “positively negative” spinners are spun, there are nine possible products, as shown in the grid below:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>12</td>
</tr>
<tr>
<td>-3</td>
<td>-12</td>
<td>-18</td>
</tr>
</tbody>
</table>

Students might notice that each number in the bottom row is equal to the opposite of the sum of the two numbers above it; for instance, in the first column, -12 is the opposite of 4 + 8. Recognizing this pattern is useful in answering the question about Jacob and Simone, because three turns with scores of 4, 8, and -12 result in 0 points. Two sets of these three scores as well as an additional score of 12 points will yield the desired result:

\[ 4 + 8 + (-12) + 4 + 8 + (-12) + 12 = 12 \]

Similarly, other sets of three scores will result in 0 points: \((-5, -10, 15)\), \((6, 12, -18)\), \((4, 6, -10)\), and \((6, 6, -12)\). Any two of these sets, combined with an additional turn of 12 points, will yield a total score of 12 points after 7 turns. The student work at the bottom of figure 2 uses this idea of balance to obtain a correct sequence for Simone’s spins.

In the context of symbolic manipulation, the idea of balance is generally thought to correspond to the procedural rule “Anything you do to one side of the equation, you must do to the other,” and it is important for students to understand this rule when developing fluency with symbols. However, a broader view of balance refers to understanding equivalence and opposites. Although the “positively negative” game helps students to acquire proficiency with positive and negative numbers, it also enables them to develop an intuitive understanding of balance.

Algebraic Thinking and a Geometry Task

THE DECREASING VOLUME TASK, TAKEN FROM Exploring Volume and Dimension (ETS 2003b), initially asks students to determine which dimension of a 5 × 7 × 15 box should be decreased by 1 unit to produce the greatest decrease in volume. Students are then asked to generalize which dimension of any rectangular box should be reduced by 1 unit to yield the greatest decrease in volume.

For the specific case of a 5 unit × 7 unit × 15 unit box, the typical solution strategy involves testing each case. Generally, students proceed as follows:

- If the 5-unit side is reduced, the new box will be 4 × 7 × 15, and have a volume of 420 cubic units.
- If the 7-unit side is reduced, the new box will be 5 × 6 × 15, and have a volume of 450 cubic units.
- If the 15-unit side is reduced, the new box will be 5 × 7 × 14, and have a volume of 490 cubic units.

From these observations, students concluded that reducing the 5-unit side would yield the greatest decrease in volume.

![Fig. 2 A student solution for the Simone and Jacob problem using the idea of balance](image-url)
Based on the exploration of a $5 \times 7 \times 15$ box, the Decreasing Volume task may appear to be solely geometric. Indeed, even the general case regarding a box of any size can be solved with geometric reasoning, as was done by the student whose work is shown in figure 3. However, the task extends to algebraic thinking because students are asked to generalize their findings. The student work in figure 4 does not use algebraic symbols, but it does use inductive reasoning to reach a conclusion. Using the previous results from the $5 \times 7 \times 15$ box, as well as considering boxes with dimensions of $3 \times 4 \times 5$ and $5 \times 2 \times 8$, the student recognizes a pattern from three specific examples. More to the point, the student’s work in figure 5 exhibits obvious algebraic thought in that the student used variables to represent and analyze the situation. The student pictured the three slices that could be removed from the box and represented their volumes with formulas using $l$, $w$, and $h$. If $h$ is the shortest dimension, the slice with dimensions $w \times l \times 1$ has the greatest volume. Therefore, the student concluded that the box with volume $lwh - wll$ has the greatest decrease from the original. This is a generalized form of the strategy used in figure 4. Since algebra is “the generalization of arithmetic,” this solution clearly demonstrates a great deal of algebraic thinking.

The conclusion in figure 4 is based on boxes of several sizes, whereas the conclusion in figure 5 is based on every possible box size, $l \times w \times h$. Consequently, even though each student receives full credit for his or her work, the explanation given by the student in figure 5 is more complete. Showing students an algebraic approach to solving problems like this may be used to introduce the power of symbolic representation. In addition, this problem can be revisited as students acquire algebraic skills and as they encounter other situations about which they are asked to generalize.

**Data Analysis in an Algebra Task**

SOMETIMES THE CONNECTION BETWEEN ALGEBRA and other Standards turns up where you would least expect it. The Odd Integers problem below was given to a group of students to determine their ability to translate word problems into symbols and to solve equations.

The sum of four consecutive odd integers is 112. What is the greatest of these four integers? (MATHCOUNTS 2003, p. 51)

One teacher who used this problem expected students to represent each of the four odd numbers with an algebraic expression, set up an equation, solve for a variable, and interpret the results. Figure 6 shows one solution that used this expected method, as well as two other successful solutions.

Solution A exhibits a strategy typical of what we might expect from a first-year algebra student, and solution B employs a guess-and-check strategy. Although equally correct, solution B relies entirely on number sense and computation. Solution C, although atypical, demonstrates number sense principles and a significant amount of algebraic thinking.
The first step (dividing by 4) finds the average of the four numbers. In stating that the “third number is 28 + 1,” the student seems to use a variable implicitly to represent an unknown quantity: the average of the four numbers is \( x \), and the third number is \( x + 1 \), using the symmetry of odd numbers on either side of the mean.

The idea of averaging is often described as an “evening out” of data: take a little from this pile and add it to that pile. By referring to averages, solution C connects algebra to data analysis. Each of the integers can be represented by an expression, an arrangement of algebra tiles, or a diagram. As figure 7 shows, the algebraic average of the four integers is \( x + 3 \): remove a square from \( x + 4 \) and add it to \( x + 2 \), and remove three squares from \( x + 6 \) and add them to \( x \). Numerically, the average of the four numbers is \( 112 \div 4 = 28 \). As a result, \( x + 3 = 28 \), which means that \( x = 25 \), and the greatest of the four integers is \( x + 6 \), or 31.

The Odd Integers problem is a valuable task for middle school students, since it invites solutions that involve multiple strands and enables teachers to highlight how one problem can be solved with various strategies. Unlike the “positively negative” game, which is a number sense task that incorporates algebraic ideas, and Decreasing Volume, which integrates algebraic concepts into a geometric task, the Odd Integers problem is foremost an algebra problem—one that might appear in a typical eighth-grade algebra course—that allows for the exploration of a topic from data analysis.

**Modifying Tasks to Promote Algebraic Thinking**

ANOTHER EXAMPLE OF THE CONNECTION BETWEEN data analysis and algebra can be seen in the Bookworms task from Analyzing Data and Making Predictions (ETS 2003a). As shown in figure 8, the task presents data collected from student surveys. Although most of the questions assess a student’s data analysis skills, question 4 asks students to pre-
dict results for the entire school population. A solution that employs algebraic thinking for question 4 of the Bookworms task is shown in figure 9. The method used in figure 10, although not strictly algebraic, also requires that students apply proportional reasoning, which is needed for many algebraic tasks.

The solutions in figures 9 and 10 are both correct, and there are advantages to showing students the value of each method. On the one hand, by not requiring a particular solution strategy, this task is valuable for expanding students’ problem-solving skills. On the other hand, the problem can be used to assess a student’s ability to translate word problems to equations and to manipulate symbols. To obtain evidence for this purpose, it may be necessary to ask a follow-up question, such as the following:

Explain how the problem might be solved using a proportion.

This minor modification makes the Bookworms task appropriate for assessing proportional reasoning, which is essential for algebraic thinking. However, substantial changes are occasionally required to make a task suitable for students. For example, the final part of the Decreasing Volume task requires students to explain the solution for any size box, but such a task may be difficult for students who have minimal experience in generalizing results. Adding the information shown in table 1 should help students identify a pattern as dimensions are changed. Using the results from table 1, the student can be asked to explain which dimension should be reduced by 1 unit to produce the greatest decrease in volume for any size box.

As shown by the last row of the table, the modification can include variables, allowing for the discovery, reinforcement, and assessment of symbolic representation and manipulation skills.

Similarly, as shown in figure 6, the Odd Integers problem can be solved using algebra, number sense, or data analysis techniques. To ensure that the task requires students to apply algebraic thinking, any of the following additions or modifications could be used:

- If $n$ is an odd integer, what expressions could be used to represent the next three odd numbers?
- Four consecutive odd integers are represented by $n$, $n+2$, $n+4$, and $n+6$, and the sum of these four numbers is 112. Solve the equation $n + (n + 2) + (n + 4) + (n + 6) = 112$ to determine the values of the four integers.
- The smallest of four consecutive odd integers is $n$, and the sum of these four numbers is 112. In the three blanks below, write expressions that

![Figure 9: A solution using algebraic reasoning for the Bookworms task](image)

![Figure 10: A nonalgebraic solution to the Bookworms task](image)
TABLE 1
Identifying a Pattern When Dimensions Are Changed

<table>
<thead>
<tr>
<th>DIMENSIONS OF BOX</th>
<th>VOLUME OF BOX WHEN SMALLEST DIMENSION IS REDUCED BY 1</th>
<th>VOLUME OF BOX WHEN MIDDLE DIMENSION IS REDUCED BY 1</th>
<th>VOLUME OF BOX WHEN LARGEST DIMENSION IS REDUCED BY 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3 \times 5 \times 7$</td>
<td>$2 \times 5 \times 7 = 70$</td>
<td>$3 \times 4 \times 7 = 84$</td>
<td>$3 \times 5 \times 6 = 90$</td>
</tr>
<tr>
<td>$4 \times 6 \times 8$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$10 \times 5 \times 6$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m \times n \times p$</td>
<td>$(m - 1) \times n \times p$</td>
<td>$m \times _____ \times p$</td>
<td>$mnp - np$</td>
</tr>
</tbody>
</table>

could be used to represent the other three odd integers in terms of $n$. Then solve the equation to find the values of the four numbers.

$$n + _____ + _____ + _____ = 112$$

When choosing tasks for the classroom, it is important to keep curricular goals in mind. Many commercially available tasks and activities contain rich contexts and problems, but not all of them align with district or state standards, and when used as performance assessments, they may not measure what was intended. When using off-the-shelf products, you may find the following checklist helpful in determining if a task is appropriate for your students as is, if the task will require some adjustment to make it appropriate, or whether it should be used at all:

- Does the task cover important content and processes?
- Do the directions clearly indicate what is expected of the student?
- Is the task capable of eliciting the best possible performance from all students?
- Will all students understand and interpret the task in the same way?
- What modifications would make this task more appropriate for my students? (Adapted from Santel-Parke and Cai 1997)

In addition to the modifications noted above, other adjustments may improve a task's usefulness in your classroom and may foster algebraic thinking. Improvements to consider are listed below:

- **Modifying the context.** Although an electrical engineering problem may contain important mathematics, students might be more interested if the context involves video games.
- **Requiring a specific strategy.** By directing students to demonstrate a particular skill, a task may better align with state and district goals and more completely assess student understanding.

- **Changing the numbers.** Using only whole numbers may make a problem more accessible to struggling students, whereas fractions and decimals may make a problem more challenging for high achievers and often more consistent with real-world situations.

**Conclusion**

AS SHOWN WITH THE DECREASING VOLUME TASK in this article, a slight adjustment can take middle school students from calculating the volume of rectangular boxes, to investigating what happens when one dimension of the box is reduced, to determining which dimension should be reduced to generate the greatest decrease in volume for any size box. Thus, students are now engaged in algebraic thinking as they generalize results! The Venn diagram in figure 11 shows how a modified version of the Decreasing
Volume task fosters algebraic thinking while addressing three Content Standards in the mathematics curriculum.

Frequently, middle-grades teachers choose tasks that can be used to foster algebraic thinking. It is important that teachers learn both to recognize the algebra that occurs naturally in tasks focusing on other strands and to turn those occurrences into meaningful learning experiences. Many middle-grades tasks contain rich content, interesting contexts, and engaging mathematics. Highlighting the algebraic opportunities that already exist in those tasks, as well as modifying tasks to ensure that they foster algebraic thinking, can go a long way toward preparing your students for the mathematical road ahead.

References


